Int J Theor Phys (2010) 49: 152–161
DOI 10.1007/s10773-009-0188-6

Thermodynamics of Black Holes in Einstein-Maxwell-Gauss-Bonnet Theory

Ritabrata Biswas · Subenoy Chakraborty

Received: 10 July 2009 / Accepted: 3 November 2009 / Published online: 12 November 2009

Abstract We study the black hole solution in Einstein-Maxwell-Gauss-Bonnet (EMGB) gravity theory with a cosmological constant in five dimension. It is a generalization of the Reissner-Nordström-de Sitter (RNdS) or RNAdS (Reissner-Nordström-Anti-de Sitter) black hole solutions (according as the cosmological constant is positive or negative) in the Einstein-Gauss-Bonnet (EGB) theory. We analyze the thermodynamic quantities of EMGB black hole and find a restriction involving the charge and the cosmological constant for the existence of an extremal black hole. Finally, Hawking-Page phase transition has been discussed for the present black hole.

Keywords Thermodynamics · Black hole · Phase transition

1 Introduction

A thermodynamical system and a black hole are beautifully interrelated by the geometry of the event horizon: Black hole temperature (known as Hawking temperature) is proportional to surface gravity on the horizon while entropy is related to the area of event horizon [1, 2] and they follow the first law of thermodynamics [3]. Although, the statistical nature of the black hole thermodynamics is completely unknown, yet the thermodynamical stability of the black hole is characterized by the sign of its heat capacity (c_v). In fact, a black hole is said to be thermodynamically unstable (as Schwarzschild black hole) if $c_v < 0$ while if c_v changes sign in the parameter space such that it diverges [4] in between then ordinary thermodynamics tells us a second order phase transition [5–7]. Further, for extremal black hole there exists a critical point and phase transition takes place from an extremal black hole to its non-extremal counter part.

R. Biswas · S. Chakraborty (✉)
Department of Mathematics, Jadavpur University, Kolkata-32, India
e-mail: schakraborty@math.jdvu.ac.in

R. Biswas
e-mail: biswas.ritabrata@gmail.com

It is well known that the Schwarzschild black hole having negative specific heat is in an unstable equilibrium with T the temperature of the heat reservoir [8]. In fact due to small fluctuations the Schwarzchild black hole will either decay (to hot flat space) by Hawking radiation or grow absorbing thermal radiation in the heat reservoir [9]. However it is possible to have a thermodynamically stable black hole having positive specific heat through a phase transition from thermal radiation [10, 11]. In this context, a mechanism of phase transition was introduced by Hawking and Page [12] showing the transition between thermal AdS space and Schwarzchild-AdS (SAdS) black hole [10, 11, 13, 14].

In this work, we check whether Hawking-Page (HP) phase transition occurs in EMGB spaces. We analyze the thermodynamical variables namely heat capacity, free energy etc for the possibility of a phase transition. Also we have studied the limiting values of the parameters involved to identify the properties of known black holes. The paper is organized as follows: Sect. 2 gives the description of 5-D EMGB black hole while in Sect. 3 thermodynamics of that black hole has been studied. The paper ends with discussion and conclusion in Sect. 4.

2 5-D Black Hole Solution in Einstein-Maxwell Theory with Gauss-Bonnet Term

The action for 5-D EMGB theory [15–18] with a cosmological term Λ has the expression in $+3$ signature (choosing $G = 1 = c$)

$$A = \int d^5x\sqrt{-g}\left[R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \alpha R_{GB}\right] \tag{1}$$

where

$$R_{GB} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2,$$

is the GB term, α is the GB-coupling parameter having dimension $(length)^2$ and is related to string tension as α^{-1}. By variational principle (i.e., $\delta S = 0$) we obtain the Einstein-Maxwell equations with GB term

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}^{(EM)} + T_{\mu\nu}^{(GB)} \tag{2}$$

where

$$T_{\mu\nu}^{(EM)} = F_{\mu\alpha}F_{\nu}^{\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

is the usual electromagnetic energy-momentum tenor and

$$T_{\mu\nu}^{(GB)} = \alpha\left[4R^{\alpha\beta}R_{\mu\alpha\nu\beta} - 2R_{\mu\alpha\beta\gamma}R_{\nu}^{\alpha\beta\gamma} + 4R_{\mu\alpha}R_{\nu}^{\alpha} - 2RR_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R_{GB}\right]$$

is the effective tensor associated with the quadratic GB-term.

For 5-D spherically symmetric space-time with metric ansatz

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega_3^2 \tag{3}$$

the non-vanishing components of the modified Einstein field equations (2) has a possible solution [16, 17]

$$B(r) = 1 + \frac{r^2}{4\alpha} - \frac{r^2}{4\alpha}\sqrt{1 + \frac{16m\alpha}{\pi r^4} - \frac{8q^2\alpha}{3r^6} + \frac{4\Lambda\alpha}{3}}. \tag{4}$$

Here the electric field is chosen along the radial direction, so that the non-vanishing components of the electromagnetic tensor ($F_{\mu\nu}$) in an orthonormal frame are

$$F_{\hat{t}\hat{r}} = -F_{\hat{r}\hat{t}} = \frac{q}{4\pi r^3}.$$

One may note that there is another possible solution for $B(r)$ with positive sign in front of the square root term in (4). But that solution never vanishes for any real r and positive α. So we consider only the negative branch solution given by (4). If α is very small then the metric coefficient $B(r)$ can be approximately written as

$$B(r) \simeq 1 - \frac{2m}{\pi r^2} + \frac{q^2}{3r^4} - \frac{r^2}{l^2} + O(\alpha) \tag{5}$$

where $l = \sqrt{\frac{6}{\Lambda}}$ is the curvature scalar.

So, in the limit $\alpha \to 0$ the above solution reduces to the usual 5D Einstein-Maxwell solution (i.e., Reissner-Nordstr$\ddot{o}$m solution) with a cosmological constant. Thus in the solution (4) the two parameters $m(> 0)$ and q are related to the mass and charge of the system. Moreover, the solution (4) is well defined with a lower bound of r at r_0 (say), i.e., the solution is valid for $r \geq r_0$ where r_0 satisfies,

$$1 + \frac{16m\alpha}{\pi r_0^4} - \frac{8q^2\alpha}{3r_0^6} + \frac{4\Lambda\alpha}{3} = 0. \tag{6}$$

This lower limit of the radial coordinate, i.e., the hyper-surface $r = r_0$ is essentially a curvature singularity and the given solution will be a black hole solution if this singular hyper surface is surrounded by the event horizon (having radius r_h such that $B(r_h) = 0$), otherwise the solution describes a naked singularity.

For extremal EMGB black hole both $B(r)$ and $\frac{dB}{dr}$ must vanish [19] at the degenerate horizon and these conditions lead to

$$\Lambda r_e^6 - 3r_e^4 + q^2 = 0 \tag{7}$$

where r_e is the radius of the event horizon for extremal black hole. This cubic equation in r_e^2 has the solution

$$r_e^2 = \frac{1}{\Lambda}\left(1 + 2\cos\frac{\theta}{3}\right) \tag{8}$$

where

$$\cos\theta = 1 - \frac{\Lambda^2 q^2}{2} \quad \text{and} \quad \Lambda^2 q^2 < 4.$$

Here, r_e does not depend on the GB parameter α so this is extremal horizon radius also for RNdS black hole. Further, at $\Lambda^2 q^2 = 4$, we have $r_e^2 = \frac{\sqrt{3}+1}{\Lambda}$ while $r_e = 0$ for $\Lambda q = 0$. Note that we could not find the extremal EMGB black hole for $\Lambda^2 q^2 > 4$ and we call it as the forbidden region for EMGB black hole. Thus $0 \leq \Lambda^2 q^2 \leq 4$ is the meaningful parameter space for EMGB black hole.

The reduced mass $M = \frac{2m}{\pi}$ defined by $B(r) = 0$ is given by

$$M(r_h, q) = 2\alpha + r_h^2 + \frac{q^2}{3r_h^2} - \frac{\Lambda r_h^4}{6} \tag{9}$$

so the mass of the extremal black hole is given by

$$M_e = M(r_e, q) = 2\alpha + \left(\frac{r_e^2}{2} + \frac{q^2}{6r_e^2}\right). \tag{10}$$

Note that if $M > M_e$ then there are more than one horizon while there will be degenerate horizon at $r = r_e$ for $M = M_e$. But for $M < M_e$, no horizon exists and we are left with a naked singularity.

Moreover, for negative cosmological constant, the solution $B(r)$ for small α takes the form

$$B(r) \simeq 1 - \frac{m}{r^2} + \frac{q^2}{r^4} + \frac{r^2}{L^2} + O(\alpha)$$

where $L^2 = -\frac{6}{\Lambda}$ is the square of the curvature radius in this case. So as $\alpha \to 0$, $q \to 0$ we have

$$B(r) = 1 - \frac{m}{r^2} + \frac{r^2}{L^2} \tag{11}$$

which is nothing but the 5D topological AdS black hole [20, 21]. The extremality condition (7) now becomes

$$\Lambda r_e^6 + 3r_e^2 - q^2 = 0$$

which has the solution

$$r_e^2 = \frac{1}{\Lambda}\left(2\cos\frac{\theta}{3} - 1\right), \qquad \cos\theta = \frac{\Lambda^2 q^2}{2} - 1, \quad 0 < \Lambda^2 q^2 < 4$$

or

$$\frac{1}{\Lambda}\left[-1 + \left\{\left(\frac{\Lambda^2 q^2}{2} - 1\right) - \Lambda q\sqrt{\frac{\Lambda^2 q^2}{4} - 1}\right\}^{\frac{1}{3}} + \left\{\left(\frac{\Lambda^2 q^2}{2} - 1\right) + \Lambda q\sqrt{\frac{\Lambda^2 q^2}{4} - 1}\right\}^{\frac{1}{3}}\right],$$
$$\Lambda^2 q^2 > 4. \tag{12}$$

Hence there is no forbidden region in this case and $\Lambda^2 q^2 = 0$ should be the limit in both cases (i.e., for positive Λ and negative Λ).

3 Thermodynamics of the EMGB Black Hole

The thermodynamics of EMGB black hole with cosmological constant is essentially an extension of the RNdS black hole (due to GB theory) in the canonical ensembles. The Hawking temperature defined as $T_H = \frac{B'(r_h)}{4\pi}$, has the explicit expression

$$T_H = \frac{(3r_h^4 - q^2 - \Lambda r_h^6)}{6\pi r_h^3 (r_h^2 + 4\alpha)}. \tag{13}$$

The other relevant thermodynamic quantities at the event horizon are the following [19, 20]:

$$\text{Bekenstein–Hawking entropy} \quad (S_{BH}) = \frac{V_3 r_h^3}{4} = \frac{\pi^2 r_h^3}{2},$$

$$\text{Electric potential}\quad (\phi)=\left(\frac{\partial M}{\partial q}\right)_S=\frac{2}{3}qr_h^{-2},$$

$$\text{Heat capacity}\quad (c_q)=T_H\left(\frac{\partial S}{\partial T_H}\right)_q$$

$$=\frac{\frac{3}{2}\pi^2 r_h^3(r_h^2+4\alpha)(3r_h^4-q^2-\Lambda r_h^6)}{[-\Lambda r_h^8-3r_h^6(1+4\Lambda\alpha)+12\alpha r_h^4+5q^2r_h^2+12\alpha q^2]},\tag{14}$$

$$\text{Free energy}\quad (F)=\frac{3V_3}{16\pi}M-T_H.S_{BH}$$

$$=\frac{V_3}{48\pi}\frac{[\frac{\Lambda}{2}r_h^8+3(1-2\alpha\Lambda)r_h^6+54\alpha r_h^4+(72\alpha^2+5q^2)r_h^2+12\alpha q^2]}{r_h^2(r_h^2+4\alpha)},$$

$$\text{ADM mass (or energy) } M_E(\text{or } E)=\frac{3V_3M}{16\pi}=F+T_HS_{BH}=\frac{3}{8}\pi M$$

where $V_3=2\pi^2$ is the volume of a unit three sphere. One may note that the ADM mass and the parameter $'m'$ are related to the black hole mass M *only by a multiplicative factor*.

We now present the above thermodynamical quantities for the limiting case $\alpha\to 0, q\to 0$. So we have,

$$\begin{aligned}
M&\to r_h^2\left(1-\frac{r_h^2}{l^2}\right),\\
T_H&\to \frac{1}{2\pi r_h}-\frac{r_h}{\pi l^2},\\
c_q&\to 3S_{BH}\left(\frac{2r_h^2-l^2}{2r_h^2+l^2}\right),\\
F&\to \frac{V_3r_h^2}{16\pi}\left(1+\frac{r_h^2}{l^2}\right).
\end{aligned}\tag{15}$$

One can check that these thermodynamical parameters are same as those of Schwarzchild-de Sitter (SdS) black hole [20].

Note that if we replace Λ by $-\Lambda$ then we obtain the thermodynamical quantities for EMGB black hole with a negative cosmological constant. Now in the limit $\alpha\to 0, q\to 0$ the thermodynamical quantities become

$$\begin{aligned}
M&\to r_h^2\left(1+\frac{r_h^2}{L^2}\right),\\
T_H&\to \frac{1}{2\pi r_h}+\frac{r_h}{\pi L^2},\\
c_q&\to 3S_{BH}\left(\frac{2r_h^2+L^2}{2r_h^2-L^2}\right),\\
F&\to \frac{V_3r_h^2}{16\pi}\left(1-\frac{r_h^2}{L^2}\right),
\end{aligned}\tag{16}$$

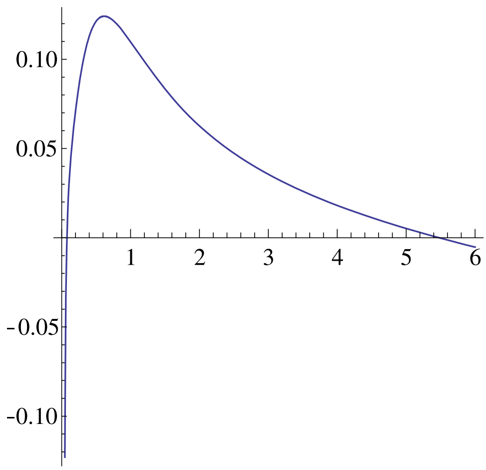
which as expected correspond to 5D Topological Anti-de Sitter (TAdS) black hole [20].

Moreover, if we choose the positive branch solution of the modified Einstein field equations (2) with the metric (3) then for negative GB coupling parameter α the solution describes a black hole with thermodynamical quantities given by (9), (13) and (14) with $-\alpha$ in the place of α. The analysis will be very similar and as before we have respectively SdS black hole and TAdS black hole as $|\alpha| \to 0$ for positive and negative Λ.

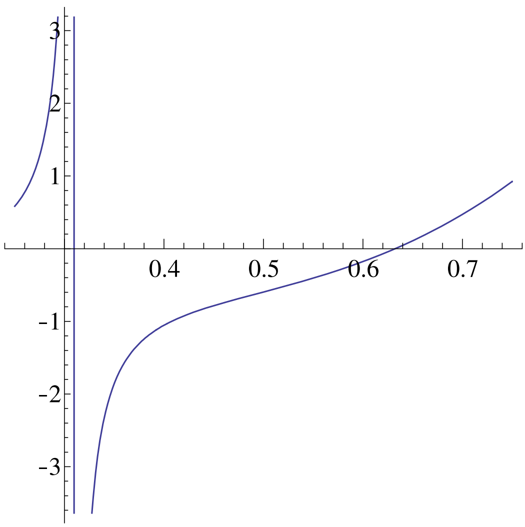
4 Discussion and Conclusion

We shall now analyze the thermodynamical quantities graphically. From Fig. 1(a) we see that T_H vanishes at $r_h = r_1$ and $r_h = r_2$ and there is a maximum at $r_h = r_m$ where $r_1 < r_m < r_2$. The graph of c_q (Fig. 1(b)) shows that for small r_h, c_q is positive and gradually blows up at

Fig. 1 (**a**), (**b**) and (**c**) show the variation of T, c_q and F with respect to r_h while (**d**) shows the variation of F with the variation of T_H

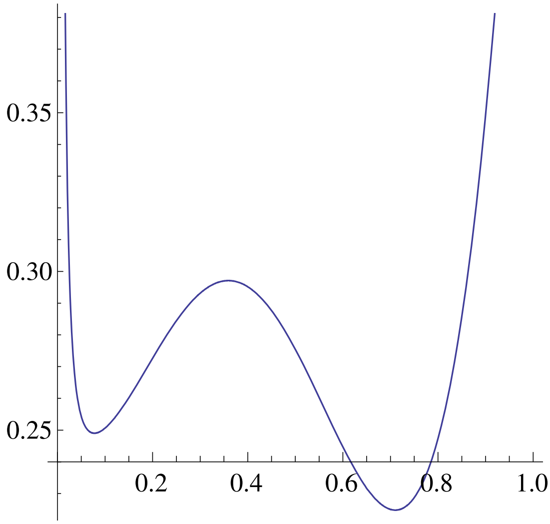


(a) $T_H - r_h$ curve for $\Lambda = 0.1$, $\alpha = 0.1$, $q = 0.01$

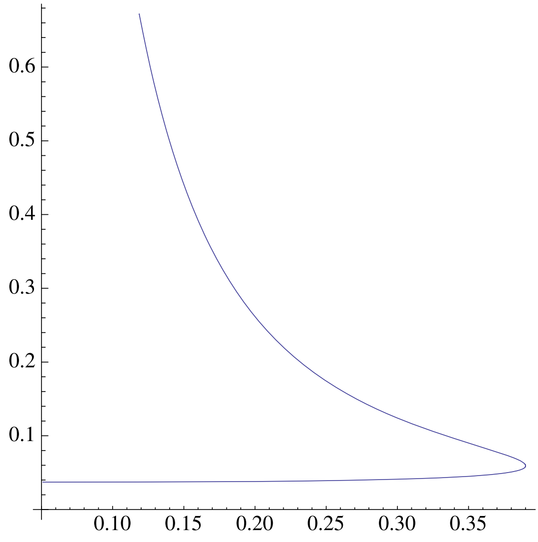


(b) $c_q - r_h$ curve for $\Lambda = 7.5$, $\alpha = 0.1$, $q = 0.01$

Fig. 1 (*Continued*)



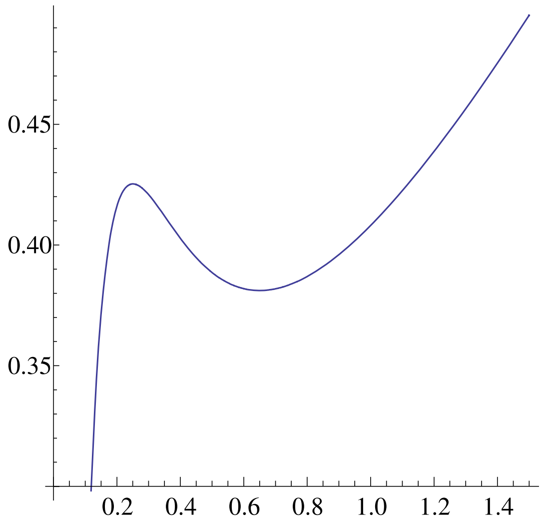
(c) $F - r_h$ curve for $\Lambda = 30$, $\alpha = 0.1$, $q = 0.01$



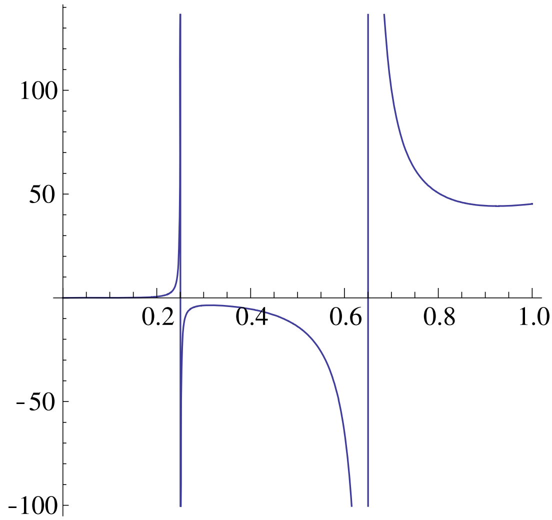
(d) $F - T_H$ curve for $\Lambda = 0.1$, $\alpha = 0.01$, $q = 0.01$

$r = r_{s_1}$ (say) and then changes sign so that c_q becomes negative. Again it gradually increases and becomes positive at $r = r_{s_2}$ (say). So at $r = r_{s_1}$, there will be a possible phase transition from stable black hole solution to a unstable one. In the limit $\alpha \to 0$, $\Lambda \to 0$, $q \to 0$ the phase transition leads to unstable Schwarzchild black hole. The graphical representation of the free energy (F) (Fig. 1(c)) against r_h shows that F remains positive having two local minimum and a local maximum. Although c_q changes sign (from positive to negative and then again positive) but F remains positive so the black hole considered here is a unstable one. Also considering local stability we say that a stable small black hole becomes a unstable intermediate black hole in $r_{s1} < r_h < r_{s_2}$ and then again becomes stable large black hole for

Fig. 2 (**a**), (**b**) and (**c**) show the variation of T, c_q and F with respect to r_h while (**d**) shows the variation of F with the variation of T_H for $\Lambda = -5$ (for the first three) and -4.5 (for the (**d**) case), $\alpha = 0.01$, $q = 0.01$



(a) $T_H - r_h$ curve

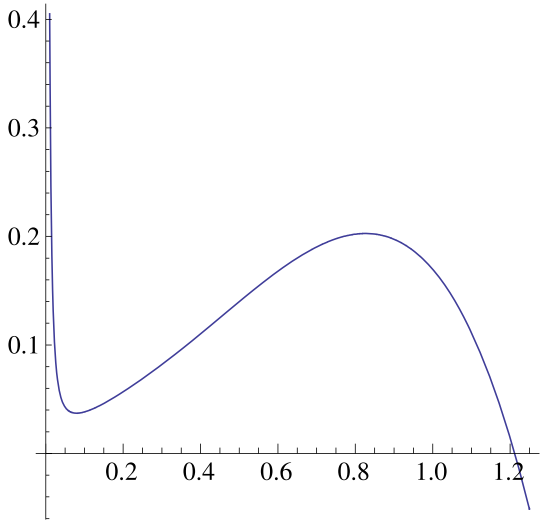


(b) $c_q - r_h$ curve

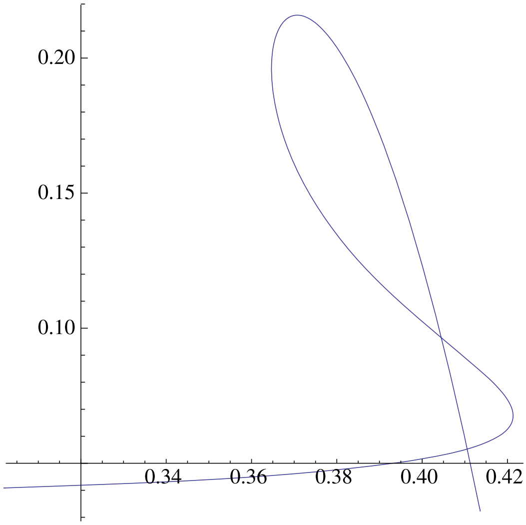
$r_h > r_{s_2}$. The variation of F with T_H (see Fig. 1(d)) shows no crossing or double point and hence no Hawking-Page phase transition is possible in this case.

When the cosmological constant is negative then in the limit $\alpha \to 0, q \to 0$ we have topological AdS black hole. The graphical representation of the thermodynamical quantities can be interpreted as follows: The Hawking temperature (in Fig. 2(a)) starts from zero at small r_h $(= r_{h_0}$, say) increases sharply reaches a local maximum (at r_{m_1}) and then decreases gradually to a local minimum (at r_{m_2}) and finally increases steadily. The graph of c_q (i.e., Fig. 2(b)) is very peculiar: It starts from small positive value and increases sharply to $+\infty$ at $r_h = r_{\infty_1}$ (say) and then changes sign and then decreases to $-\infty$ at $r_h = r_{\infty_2}$ (say) and becomes positive. The free energy (F) (in Fig. 2(c)) however, starts from a very large value falls to local minimum at $r_h = r_{fm_1}$ and then increases gradually to a local maximum at

Fig. 2 (*Continued*)



(c) $F - r_h$ curve



(d) $F - T_H$ curve

$r_h = r_{fm_2}$ and finally decreases and becomes negative at $r_h = r_{f0}$. As c_q changes from positive to negative and then again positive in going through singularities of c_q so two phase transitions are possible − one from small stable black hole to unstable intermediate black hole and then next transition from unstable intermediate black hole to stable large black hole. The second phase transition is from ($c_q < 0$, $F > 0$) to ($c_q > 0$, $F < 0$) and may be identified as Hawking-Page phase transition. Finally, the graphical representation (Fig. 2(d)) of F against T_H also shows two possible phase transitions at the two double points.

Acknowledgement The work is done during a visit to IUCAA. The authors are thankful to IUCAA for warm hospitality and facilities of doing research.

References

1. Hawking, S.W.: Commun. Math. Phys. **43**, 199 (1975)
2. Bekenstein, J.D.: Phys. Rev. D **7**, 2333 (1973)
3. Bardeen, J.M., Carter, B., Hawking, S.W.: Commun. Math. Phys. **31**, 161 (1973)
4. Hut, P.: Mon. Not. R. Astron Soc. **180**, 379 (1977)
5. Davies, P.C.W.: Proc. R. Soc. Lond. A **353**, 499 (1977)
6. Davies, P.C.W.: Rep. Prog. Phys. **41**, 1313 (1977)
7. Davies, P.C.W.: Class. Quantum Gravity **6**, 1909 (1989)
8. Gross, D.J., Perry, M.J., Yaffe, L.G.: Phys. Rev. D **25**, 330 (1982)
9. York, J.W.: Phys. Rev. D **33**, 2092 (1986)
10. Myung, Y.S.: Phys. Rev. D **77**, 104007 (2008)
11. Myung, Y.S.: Phys. Lett. B. **645**, 369 (2007)
12. Hawking, S.W., Page, D.N.: Commun. Math. Phys. **87**, 577 (1983)
13. Brown, J.D., Creighton, J., Mann, R.B.: Phys. Rev. D **50**, 6394 (1994)
14. Witten, E.: Adv. Theor. Math. Phys. **2**, 505 (1998)
15. Boulware, D.G., Deser, S.: Phys. Rev. Lett. **55**, 2656 (1985)
16. Wiltshire, D.L.: Phys. Lett. B. **169**, 36 (1986)
17. Wiltshire, D.L.: Phys. Rev. D **38**, 2445 (1988)
18. Thibeault, M., Simeone, C., Eirod, E.F.: Gen. Relativ. Gravit. **38**, 1593 (2006)
19. Myung, Y.S., Kim, Y.W., Park, Y.J.: Phys. Rev. D. **78**, 084002 (2008)
20. Myung, Y.S.: Class. Quantum Gravity **21**, 1279 (2004)
21. Birmingham, D.: Class. Quantum Gravity **16**, 1197 (1999)